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Viscoelastic boundary layer flow and heat transfer over an exponential stretching sheet

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Abstract

Viscoelastic boundary layer flow and heat transfer over an exponential stretching continuous sheet have been examined in this paper. Approximate analytical similarity solution of the highly non-linear momentum equation and confluent hypergeometric similarity solution of the heat transfer equation are obtained. Accuracy of the analytical solution for stream function is verified by numerical solutions obtained by employing Runge–Kutta fourth order method with shooting. These solutions involve an exponential dependent of stretching velocity, prescribed boundary temperature and prescribed boundary heat flux on the flow directional coordinate. The effects of various physical parameters like viscoelastic parameter, Prandtl number, Reynolds number, Nusselt number and Eckert number on various momentum and heat transfer characteristics are discussed in detail in this work.

Keywords: Viscoelastic fluid; Boundary layer flow; Exponential stretching surface; Heat transfer and non-isothermal boundary

1. Introduction

Numerous applications of viscoelastic fluids in several industrial manufacturing processes have led to renewed interest among researchers to investigate viscoelastic boundary layer flow over a stretching plastic sheet (Rajagopal et al. [1,2], Dandapat and Gupta [3], Rollins and Vajravelu [4], Andersson [5], Lawrence and Rao [6], Char [7] and Rao [8]). Some of the typical applications of such study are polymer sheet extrusion from a dye, glass fiber and paper production, drawing of plastic films etc. A great deal of literature is available including those cited above on the two-dimensional viscoelastic boundary layer flow over a stretching surface where the velocity of the stretching surface is assumed linearly proportional to the distance from a fixed origin. However, Gupta and Gupta [9] have pointed out that realistically stretching of the sheet may not necessarily be linear. This situation was dealt by Kumaran and Ramanaiah [10] in their work on boundary layer flow over a quadratic stretching sheet. But their work was confined to the viscous fluid flow over stretching sheet.

One of the important aspects in this theoretical study is the investigation of heat transfer processes. This is due to the fact that the rate of cooling influences a lot to the quality of the product with desired characteristics. In view of this Ali [11] investigated thermal boundary layer by considering a power law stretching surface. A new dimension has been added in this investigation by Elbashbeshy [12] who examined the flow and heat

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transfer characteristics by considering exponentially stretching continuous surface. Elbashbeshy [12] considered an exponential similarity variable and exponential stretching velocity distribution on the coordinate considered in the direction of stretching. However, the works of Ali [11] and Elbashbeshy [12] are confined to the study of viscous fluid flow only.

Since, in reality most of the fluids considered in industrial applications are more non-Newtonian in nature, specially of viscoelastic type than viscous type, we extend the work of Elbashbeshy [12] to viscoelastic fluid flow and heat transfer. Approximate analytical similarity solutions are obtained for velocity distribution by transforming highly non-linear differential equation into Riccati type and then solving this sequentially. Similarity solution for temperature is obtained in the form of confluent hypergeometric function for non-isothermal boundary conditions of both the types (1) prescribed surface temperature (PST) of exponential order and (2) prescribed boundary heat flux (PHF) of exponential order. The aim of the article is to analyse the effect of various physical parameters like viscoelastic parameter, Prandtl number, Reynolds number, Nusselt number and Eckert number on various momentum and heat transfer characteristics of boundary layer flow of viscoelastic second-order fluid over an exponential stretching continuous surface.

2. Formulation of the problem

The constitutive equation satisfied by second-order fluid was given by Coleman and Noll [13], following the postulates of gradually fading memory, as

$$T = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 \tag{2.1}$$

where T is the Cauchy stress tensor, $-pI$ is the spherical stress due to constraint of incompressibility, μ is the dynamics viscosity, α_1, α_2 are the material moduli. A_1 and A_2 are the first two Rivlin–Ericksen tensors and they are defined as

$$A_1 = (\mathrm{grad}\, q) + (\mathrm{grad}\, q)^{\mathrm{T}} \tag{2.2}$$

$$A_2 = \frac{\mathrm{d}A_1}{\mathrm{d}t} + A_1(\mathrm{grad}\, q) + (\mathrm{grad}\, q)^{\mathrm{T}} \cdot A_1 \tag{2.3}$$

The model Eq. (2.1) was derived by considering up to second-order approximation of retardation parameter. Dunn and Fosdick [14] have shown that this model equation is invariant under transformation and thereby they concluded this model as an exact model in which the material moduli must satisfy the restrictions:

$$\mu \geqslant 0, \quad \alpha_1 \geqslant 0, \quad \alpha_1 + \alpha_2 = 0 \tag{2.4}$$

The fluid modeled by Eq. (2.1) with the relationship (2.4) is compatible with the thermodynamics. The third relation is the consequence of satisfying the Clausius–Duhem inequality by fluid motion and the second relation arises due to the assumption that specific Helmholtz free energy of the fluid takes its minimum values in equilibrium. But recent experimental results for most of the non-Newtonian fluids which assumed to be second-order have contradicted the above relations of Eq. (2.4). Fosdick and Rajagopal [15] have shown, by using the data reduction from experiments, that in the case of a second-order fluid the following relation should holds.

$$\mu \geqslant 0, \quad \alpha_1 \leqslant 0, \quad \alpha_1 + \alpha_2 \neq 0 \tag{2.5}$$

They also found that the fluids modeled by Eq. (2.1) with the relationship (2.5) exhibit some anomalous behaviour. A detail review on this controversial issue concerning the status of fluid is recorded in the work of Dunn and Rajagopal [16]. Now, generally the fluid satisfied the model Eq. (2.1) with $\alpha < 0$ is termed as second- order fluid and with $\alpha > 0$ is termed as second grade fluid (Rajagopal et al. [2]). Although second-order fluid, obeying model Eq. (2.1) with $\alpha_1 < \alpha_2$, $\alpha_1 < 0$, exhibits some undesirable instability characteristics (Fosdick and Rajagopal [15]) the second-order approximation is valid at low shear rate (Rajagopal et al. [2]).

A steady state two-dimensional boundary layer flow of incompressible second-order viscoelastic fluid over a stretching sheet has been considered for investigation. Boundary is assumed to be moving axially with a velocity of exponential order in axial distance by the application of two equal and opposite forces along x-axis keeping the origin fixed and generating the boundary layer type of flow. We take into account of frictional heating due to viscous dissipation as the fluid considered for analysis is of non-Newtonian type. The governing boundary layer equations for momentum and heat transfer in such flow situations (Cortell [17] and Rajagopal et al. [1]), in the usual form, are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.6}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \gamma\frac{\partial^2 u}{\partial y^2} - k_0\left\{u\frac{\partial^3 u}{\partial x \partial y^2} + v\frac{\partial^3 u}{\partial y^3} - \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\right\} \tag{2.7}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho c_p}\left(\frac{\partial u}{\partial y}\right)^2 \tag{2.8}$$

Here u and v are the velocity components in x and y direction respectively, γ is the kinematic coefficient of viscosity, $k_0 = \frac{-\alpha_1}{\rho}$ is the elastic parameter. Hence, in the case of second-order fluid flow k_0 takes positive value as α_1 takes negative value, $\alpha = \frac{k}{\rho c_p}$ is the thermal diffusivity, k is the thermal conductivity and other quantities have their usual meanings. In deriving Eq. (2.7) it is assumed that the normal stress is of the same order of magnitude as that of the shear stress, in addition to usual boundary layer approximations. Eq. (2.8) is the thermal boundary layer equation, which takes into account the viscous dissipation (the last term in Eq. (2.8)). However, we assume that the fluid possesses strong viscous property in comparison with the elastic property. With this assumption we neglect the contribution of heat due to elastic deformation.

2.1. Boundary conditions on velocity

For the present physical problem, where the stretching of the boundary surface is assumed to be such that the flow directional velocity is of exponential order of the flow directional coordinate, we employ the following boundary conditions (Elbashbeshy [12])

$$\begin{aligned} &u = U_w(x) = U_0 \exp\left(\frac{x}{l}\right), \quad v = 0 \qquad \text{at } y = 0 \\ &u = 0, \quad u_y = 0 \qquad \text{as } y \to \infty \end{aligned} \tag{2.9}$$

Here U_0 is a constant, l is the reference length and the suffix y represents differentiation with respect to y. It is to note that the first three boundary conditions prescribed by Eq. (2.9) are not sufficient to solve the problem uniquely. In this regard, let us have a glimpse on the existing available literature. A critical review on the boundary conditions and the existence and uniqueness of the solution have been given by Rajagopal [18]. Most of the available literature on boundary layer flow of a viscoelastic fluid over linearly stretching sheets deal with the three boundary conditions on velocity, which are one less than the number required to solve the problem uniquely (Rajagopal et al. [1,2], Rollins and Vajravelu [4], Andersson [5], Cortell [17] and Mahapatra and Gupta [19]). The augmentation of the boundary condition has also been discussed in the work of Rajagopal and Gupta [20]. Troy et al. [21] derived unique solution of the problem containing exponential terms of similarity variable. Later Chang [22] showed that the solution was not unique and derived another closed form of solution. Subsequently, Lawrence and Rao [6] presented a general method and derived both the non-unique solutions. Among all the solutions the solution given by Troy et al. [21] containing exponential term is physically realistic (Lawrence and Rao [6]) as slightly elastic fluid (assigning small value of elastic parameter in the equation) produces a boundary layer only slightly altered in its dimensions from the viscous one.

In view of the above discussions on boundary conditions we present in the next section the physically realistic sequential similarity solutions of viscoelastic boundary layer Eq. (2.7).

3. Solution of the momentum equation

Eq. (2.7) may be rewritten in terms of stream function $\psi(x,y)$ which satisfies the equation of continuity (2.6). Hence

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \tag{3.1}$$

Stream function $\psi(x,y)$ is defined as

$$\psi(x,y) = \sqrt{2\gamma l U_0} f(\eta) \exp\left(\frac{x}{2l}\right) \tag{3.2}$$

$$\eta = y\sqrt{\frac{U_0}{2\gamma l}} \exp\left(\frac{x}{2l}\right) \tag{3.3}$$

Here f is the dimensionless stream function and η is the similarity variable. Substitution of Eq. (3.2) in Eq. (2.7) results in a fourth order non-linear ordinary differential equation of the form

$$2f_\eta^2 - ff_{\eta\eta} = f_{\eta\eta\eta} - k_1^*\left[3f_\eta f_{\eta\eta\eta} - \frac{1}{2}ff_{\eta\eta\eta\eta} - \frac{3}{2}f_{\eta\eta}^2\right] \tag{3.4}$$

where $k_1^* = \frac{k_0 U_w}{\gamma l}$ is the dimensionless viscoelastic parameter.

The corresponding boundary conditions on f are of the form

$$\begin{aligned} &f = 0, \quad f_\eta = 1 \qquad \text{at } \eta = 0 \\ &f_\eta = 0, \quad f_{\eta\eta} = 0 \qquad \text{as } \eta \to \infty \end{aligned} \tag{3.5}$$

Integrating Eq. (3.4), we obtain

$$\begin{aligned} &f_{\eta\eta} + ff_\eta \\ &\quad = -S + \int_0^\eta \left[3f_\eta^2 + k_1^*\left\{3f_\eta f_{\eta\eta\eta} - \frac{1}{2}ff_{\eta\eta\eta\eta} - \frac{3}{2}f_{\eta\eta}^2\right\}\right] \mathrm{d}\eta \end{aligned} \tag{3.6}$$

where $S = -f''(0)$.

For $\eta \to \infty$, we get

$$S = \int_0^\eta \left[3f_\eta^2 + k_1^*\left\{3f_\eta f_{\eta\eta\eta} - \frac{1}{2}ff_{\eta\eta\eta\eta} - \frac{3}{2}f_{\eta\eta}^2\right\}\right] \mathrm{d}\eta \tag{3.7}$$

Integrating Eq. (3.6) once more, we get

$$\begin{aligned} f_\eta + \frac{1}{2}f^2 = 1 - S\eta + \int_0^\eta \Bigg[\int_0^{\eta_2} 3f_{\eta_1}^2 + k_1^*\Bigg(3f_{\eta_1} f_{\eta_1\eta_1\eta_1} \\ - \frac{1}{2}ff_{\eta_1\eta_1\eta_1\eta_1} - \frac{3}{2}f_{\eta_1\eta_1}^2\Bigg)\mathrm{d}\eta_1\Bigg]\mathrm{d}\eta_2 \end{aligned} \tag{3.8}$$

This equation may be solved by substituting suitable zero-order approximation $f_\eta^{(0)}(\eta)$ for $f_\eta(\eta)$ on the R.H.S. Hence the solution procedure is reduced to the sequential solutions of the Riccati-type equation of the form

$$f_\eta^{(n)} + \frac{1}{2}f^{(n)^2} = \mathrm{R.H.S}\left(f_\eta^{(n-1)}, f_{\eta\eta}^{(n-1)}, f_{\eta\eta\eta}^{(n-1)}, f_{\eta\eta\eta\eta}^{(n-1)}\right) \tag{3.9}$$

We assume zero-order approximation of $f_\eta(\eta)$ as

$$f_\eta^{(0)}(\eta) = \exp(-S_0\eta) \tag{3.10}$$

which satisfies the condition at infinity.

Integrating the expression in Eq. (3.10) we get

$$f^{(0)}(\eta) = \frac{1-\exp(-S_0\eta)}{S_0} \tag{3.11}$$

Now we substitute all the derivatives of zero-order approximation $f^{(0)}(\eta)$ into R.H.S of Eq. (3.8) and assume that first-order iteration $f^{(1)}(\eta)$ on the L.H.S. of Eq. (3.8) satisfies the boundary conditions of (3.5). Hence we get

$$S_0 = \sqrt{\frac{3}{2(1-k_1^*)}}, \quad f_{\eta\eta}^{(0)}(0) = -S_0 \tag{3.12}$$

Here, the equation for first-order iteration $f^{(1)}(\eta)$ takes the form

$$f_\eta^{(1)} + \frac{1}{2} f^{(1)^2} = 1 + \frac{(3+k_1^* S_0^2)}{4S_0^2}(e^{-2S_0\eta}-1) + \frac{k_1^*}{2}(e^{-S_0\eta}-1) \tag{3.13}$$

which is a non-linear Riccati type equation and can be solved for $f^{(1)}(\eta)$ as a function of confluent hypergeometric function. However, we take zero-order approximation $f^{(0)}(\eta)$ for solving the energy equation in the next section. This solution would enable us to obtain analytical solution of energy equation in the form of confluent hypergeometric function.

The dimensionless skin-friction coefficient C_f is expressed

$$\begin{aligned} C_f &= -\frac{\left(\gamma\frac{\partial u}{\partial y} - k_0\left\{u\frac{\partial^2 u}{\partial x\partial y} - 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right\}\right)}{U_0^2\exp\left(\frac{2x}{l}\right)} \quad \text{at } y=0 \\ &= \frac{S_0}{\sqrt{2Re}}\left[1-\frac{7}{2}k_1^*\right] \end{aligned} \tag{3.14}$$

where $Re = \dfrac{U_w l}{\gamma}$ is the Reynolds number.

4. Solution of the heat transfer equation

In order to solve the temperature Eq. (2.8) we consider two general cases of non-isothermal temperature boundary conditions, namely:

(A) Boundary with prescribed exponential order surface temperature (PST) and
(B) Boundary with prescribed exponential order heat flux (PHF).

4.1. Case A: prescribed exponential order surface temperature (PST)

In PST case we employ the following surface boundary conditions on temperature

$$\begin{aligned} &T = T_w = T_\infty + T_0\exp\left(\frac{v_0 x}{2l}\right) \quad \text{at } y=0 \\ &T = T_\infty \quad \text{as } y\to\infty \end{aligned} \tag{4.1}$$

where v_0 and T_0 are the parameters of temperature distribution on the stretching surface and T_∞ is the temperature for away from the stretching sheet.

In order to obtain similarity solution for temperature we define dimensionless temperature variable as follows:

$$\theta(\eta) = \frac{T-T_\infty}{T_w-T_\infty}$$

where $T_w - T_\infty$ is given by Eq. (4.1). With this the dimensional energy equation (2.8) takes the following non-dimensional form.

$$\theta_{\eta\eta} + Prf\theta_\eta - Prv_0 f_\eta\theta = -PrEf_{\eta\eta}^2 \tag{4.2}$$

where $Pr = \frac{\gamma}{\alpha}$ is the Prandtl number and $E = \frac{U_0^2}{c_p T_0}\left(\frac{U_w}{U_0}\right)^{\frac{4-v_0}{2}}$ is the Eckert number Boundary conditions (4.1) of temperature, in non-dimensional form, are

$$\begin{aligned} &\theta(0) = 1 \\ &\theta(\infty) = 0 \end{aligned} \tag{4.3}$$

We proceed to solve Eq. (4.2) by using zero-order approximations of f and f_η. Further we introduce the new variable

$$\xi = -\frac{Pr}{S_0^2}\exp(-S_0\eta) \tag{4.4}$$

Substitution of Eq. (4.4) in Eqs. (4.2) and (4.3) has led to the following boundary value problem.

$$\xi\theta_{\xi\xi} + (1-Pr^*-\xi)\theta_\xi + v_1\theta = \frac{-ES_0^2}{Pr^*}\xi \tag{4.5}$$

$$\begin{aligned} &\theta(\xi) = 1 \quad \text{at } \xi = -Pr^* \\ &\theta(\xi) = 0 \quad \text{at } \xi = 0 \end{aligned} \tag{4.6}$$

where

$$Pr^* = \frac{Pr}{S_0^2} \text{ is the modified Prandtl number} \tag{4.7}$$

The solution of (4.5) is assumed in the form

$$\theta(\xi) = \theta_c(\xi) + \theta_p(\xi) \tag{4.8}$$

where $\theta_c(\xi)$ is the complementary solution and $\theta_p(\xi)$ stands for particular integral. Making use of the boundary conditions (4.6) we obtain complementary solution of Eq. (4.5) in the following form of confluent hypergeometric function

$$\theta_c(\xi) = A_1 \xi^{Pr^*} M(Pr^* - v_0, Pr^* + 1, \xi) \tag{4.9}$$

Closed form particular solution exists if only we choose $v_0 = 2$ and that is obtained as

$$\theta_p(\xi) = \frac{-ES_0^2}{2Pr^*(2 - Pr^*)} \xi^2 \tag{4.10}$$

Making use of the boundary conditions of Eq. (4.6) and rewriting the solution in variable η, we get

$$\begin{aligned}\theta(\eta) &= \theta_c(\eta) + \theta_p(\eta) \\ &= \frac{(1 - C_1)e^{-S_0 Pr^* \eta} M(Pr^* - 2, Pr^* + 1, -Pr^* e^{-S_0 \eta})}{M(Pr^* - 2, Pr^* + 1, -Pr^*)} \\ &\quad + C_1 e^{-2S_0 \eta}\end{aligned} \tag{4.11a}$$

where

$$C_1 = \frac{-ES_0^2 Pr^*}{2(2 - Pr^*)} \tag{4.11b}$$

where Kummer's function M is defined by

$$M(a_0, b_0, z) = 1 + \sum_{n=1}^{\infty} \frac{(a_0)_n z^n}{(b_0)_n n!}$$

$$\begin{aligned}(a_0)_n &= a_0(a_0 + 1)(a_0 + 2) \cdots (a_0 + n - 1) \\ (b_0)_n &= b_0(b_0 + 1)(b_0 + 2) \cdots (b_0 + n - 1)\end{aligned} \tag{4.12}$$

Dimensionless wall temperature gradient $\theta_\eta(0)$ is obtained as:

$$\begin{aligned}\theta_\eta(0) = (1 - C_1) S_0 Pr^* \left[\left(\frac{Pr^* - 2}{Pr^* + 1} \right) \frac{M(Pr^* - 1, Pr^* + 2, -Pr^*)}{M(Pr^* - 2, Pr^* + 1, -Pr^*)} - 1 \right] \\ - 2C_1 S_0\end{aligned} \tag{4.13}$$

It is convenient to analysis heat transfer by means of dimensionless number of temperature gradient, known as Nusselt number. The Nusselt number Nu in the present case is derived as

$$Nu = \frac{x}{(T_w - T_\infty)} \left(\frac{\partial T}{\partial y} \right)_{y=0} = \theta_\eta(0) \sqrt{\frac{x}{2l}} \sqrt{Re_x} \tag{4.14}$$

where Re_x is the local Reynolds number and it is defined as

$$Re_x = \frac{U_w x}{\gamma}$$

4.2. Case B: prescribed exponential order power law heat flux (PHF)

In this heating process we employ the following prescribed exponential law heat flux boundary conditions.

$$\begin{aligned}&-k\left(\frac{\partial T}{\partial y} \right)_w = T_1 \exp\left(\frac{v_1 + 1}{2l} \right) x \quad \text{at } y = 0 \\ &T \to T_\infty \quad \text{as } y \to \infty\end{aligned} \tag{4.15}$$

where v_1 and T_1 are the parameters of temperature distribution on the stretching surface. In order to obtain similarity solution for temperature we define dimensionless temperature variable in PHF case as follows

$$g(\eta) = \frac{T - T_\infty}{\frac{T_1}{k} \sqrt{\frac{2\gamma l}{U_0}} \exp\left(\frac{v_1 x}{2l} \right)} \tag{4.16}$$

With this dimensionless variable and Eqs. (3.1)–(3.3), the temperature Eq. (2.8) takes the form

$$g_{\eta\eta} + Prfg_\eta - Prv_1 f_\eta g = -PrEf_{\eta\eta}^2 \tag{4.17}$$

where

$$E = \frac{U_0^2 k}{c_p T_1 \sqrt{\frac{2\gamma l}{U_0}}} \left(\frac{U_w}{U_0} \right)^{\frac{4 - v_1}{2}} \quad \text{and} \quad Pr = \frac{\gamma}{\alpha}$$

Here Pr is the Prandtl number and E is the Eckert number.

Boundary conditions on non-dimensional temperature are

$$\begin{aligned}g_\eta(0) &= -1 \\ g(\infty) &= 0\end{aligned} \tag{4.18}$$

Eq. (4.17) is the same form as Eq. (4.2). However, the first boundary condition of Eq. (4.3) differs with that of Eq. (4.18). Following the same procedure as described in the PST case and making use of the boundary conditions (4.18) we derive the solution for $g(\eta)$ in the following form of confluent hypergeometric function.

$$\begin{aligned}g(\eta) = C_2 e^{-S_0 Pr^* \eta} M[Pr^* - 2, Pr^* + 1, -Pr^* e^{-S_0 \eta}] \\ + C_1 e^{-2S_0 \eta}\end{aligned} \tag{4.19}$$

where

$$C_2 = \frac{(1 - 2C_1 S_0)}{S_0 Pr^* M(Pr^* - 2, Pr^* + 1, -Pr^*) - \frac{(Pr^* - 2)}{(Pr^* + 1)} Pr^* S_0 M(Pr^* - 1, Pr^* + 2, -Pr^*)}$$

and C_1 is given by Eq. (4.11b).

Dimensionless wall temperature $g(0)$ is obtained as

$$g(0) = C_2 M(Pr^* - 2, Pr^* + 1, -Pr^*) + C_1 \quad (4.20)$$

The expression for dimensional wall temperature is

$$T_w = T_\infty + \frac{T_1}{k}\sqrt{\frac{2\gamma l}{U_0}}\exp\left(\frac{x}{l}\right)g(0) \quad (4.21)$$

5. Results and discussion

Momentum and heat transfers in a boundary layer viscoelastic fluid flow over an exponentially stretching impermeable sheet have been investigated in this paper. The highly non-linear partial differential equations characterising flow and heat transfer have been converted to a set of non-linear ordinary differential equations by applying suitable similarity transformations. Sequential solutions of the transformed momentum equation are obtained by solving the non-linear Riccati type equation analytically. The zero-order approximate solution for dimensionless stream function f has been obtained analytically which satisfies all the boundary conditions. First-order approximate solution of f also can be derived analytically in the form of confluent hypergeometric functions. However numerical solutions for f and f_η, using Runge–Kutta fourth order method with shooting, match very well in the region which is very close to the boundary with the solution of zero-order (Fig. 1). Hence, we consider zero-order approximate solutions of f and obtain the exact analytical solutions of the heat transfer equation in the form of confluent hypergeometric functions. All these solutions involve an exponential dependence of (i) the similarity variable η (ii) stretching velocity U_w and (iii) wall temperature distribution T_w on the coordinate along the direction of stretching.

In order to have some insight of the flow and heat transfer characteristics, results are plotted graphically for typical choice of physical parameters in Figs. 2–5 and Tables 1 and 2. Velocity distributions $f^0_\eta(\eta)$ for different values of viscoelastic parameter k_1^* are shown in Fig. 2. From this figure we notice that effect of viscoelastic parameter k_1^* is to decrease velocity throughout the boundary layer flow field which is quite obvious. Fig. 3 demonstrates the graph of non-dimensional skin-friction parameter C_f vs. viscoelastic parameter k_1^* for different values of Reynolds number Re. From this figure we observe that the increase of non-dimensional viscoelastic parameter k_1^* leads to the decrease of skin-friction parameter C_f. This is due to the fact that elastic property

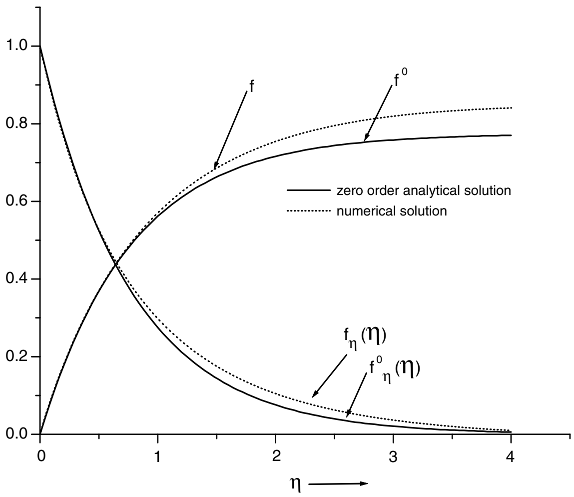

Fig. 1. Profiles for $f(\eta)$ and $f(\eta)_\eta$ obtained from numerical as well as analytical method when $k_1^* = 0.1$.

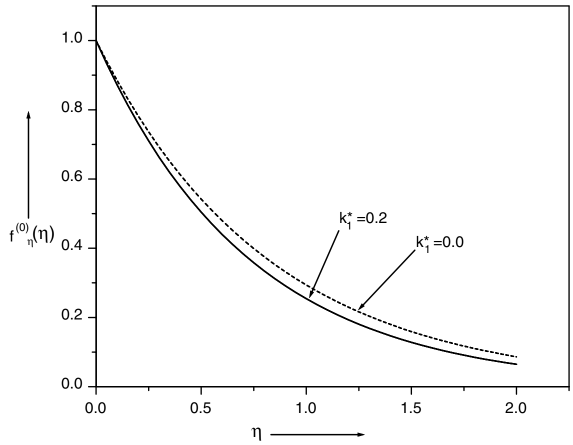

Fig. 2. Velocity profile $f_\eta^{(0)}(\eta)$ for different values of viscoelastic parameter k_1^*.

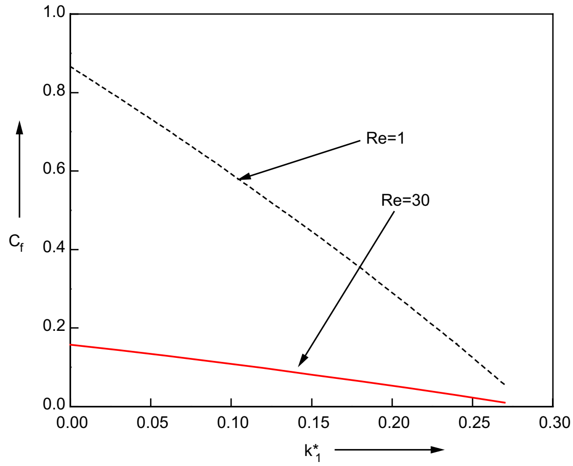

Fig. 3. Graph of skinfriction parameter C_f vs. viscoelastic parameter k_1^* for differentvalues of local Reynolds number Re.

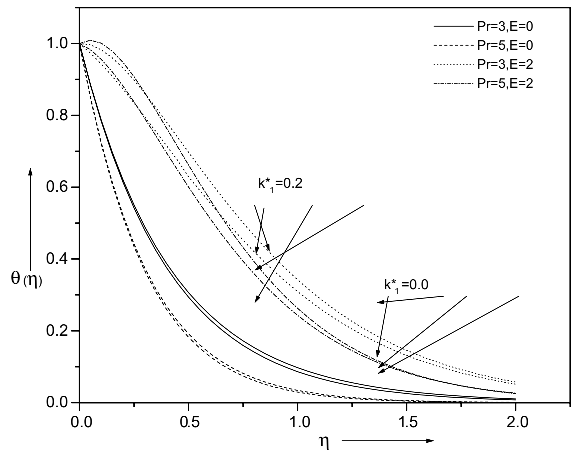

Fig. 4. Dimensionless temperature profile $\theta(\eta)$ for various values of Prandtl number Pr and Eckert number E in PST case.

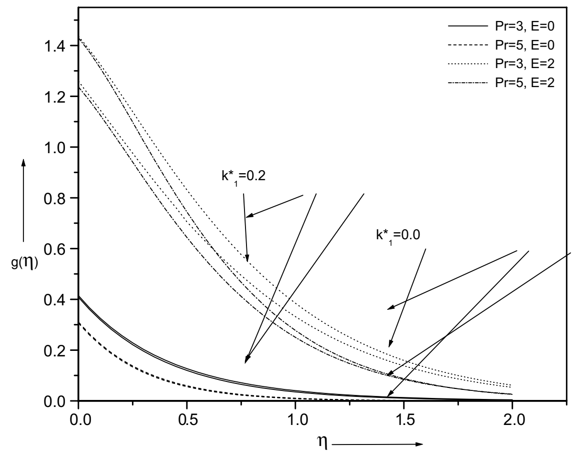

Fig. 5. Dimensionless temperature profile $g(\eta)$ for various values of Prandtl number Pr and Eckert number E in PHF case.

Table 1
Wall temperature gradient $-\theta_\eta(0)$ in PST case for different values of Prandtl number Pr, Eckert number E and viscoelastic parameter k_1^*

k_1^*	Pr	E	$-\theta_\eta(0)$
10^{-9}	3	0	2.449
0.2			2.410
10^{-9}	5		3.257
0.2			3.219
10^{-9}	3	2	0.363
0.2			−0.053
10^{-9}	5		0.227
0.2			−0.385

Table 2
Wall temperature $g(0)$ in PHF case for different values of Prandtl number Pr, Eckert number E and viscoelastic parameter k_1^*

k_1^*	Pr	E	$g(0)$
10^{-9}	3	0	0.408
0.2			0.415
10^{-9}	5		0.307
0.2			0.311
10^{-9}	3	2	1.259
0.2			1.437
10^{-9}	5		1.237
0.2			1.430

in viscoelastic fluid reduces the frictional force. This result may have great significance in polymer proceeding industry, as the choice of higher order viscoelastic fluid would reduce the power consumption for stretching the boundary sheet. We obtain the similar effect of Reynolds number on the skin-friction coefficient as reduction of viscosity of the fluid result in the decrease of frictional force or drag force.

The effect of Prandtl number Pr on heat transfer may be analysed from Figs. 4 and 5 in PST and PHF cases respectively. These graphs reveal that the increase of Prandtl number Pr results in the decrease of temperature distribution at a particular point of the flow region. This is because there would be a decrease of the thermal boundary layer thickness with the increase of values of Prandtl number Pr. The increase of Prandtl number means slow rate of thermal diffusion. It is obvious that the wall temperature distribution is at unity on the wall in PST case for all values of Pr, E and k_1^*. However, it may be other than the unity in the PHF case due to adiabatic temperature boundary condition. The effect of increasing the values of viscoelastic parameter k_1^* is seen to increase the temperatures distribution in the flow region. This is in conformity with the fact that increase of non-Newtonian viscoelastic parameter leads to the increases of thermal boundary layer thickness. The results of PHF cases are qualitatively similar to that of PST case but quantitatively they are different. The graphs reveal that the effect of increasing the values of Eckert number E is to increase temperature distribution $\theta(\eta)$ in the flow region in both the cases of PST and PHF. This behaviour of temperature enhancement occurs as heat energy is stored in the fluid due to frictional heating.

Numerical values of wall temperature gradient $-\theta_\eta(0)$ in PST case for different values of Prandtl number Pr, Eckert number E and viscoelastic parameter k_1^* are recorded in Table 1. The table reveals that the increase of the values of Prandtl number Pr for $E = 0$ and ($E \neq 0$) results in the increase of the values of wall temperature gradient $-\theta_\eta(0)$. We notice that wall temperature gradient $-\theta_\eta(0)$ is decreased by increasing the values of the viscoelastic parameter k_1^*. The effect of viscous dissipation ($E \neq 0$) is to reduce the wall temperature gradient $-\theta_\eta(0)$. Hence, by increasing the values of viscoelastic parameter k_1^* and Eckert number E we can control heat transfer considerably. Significant increase of Eckert number might reverse the direction of heat transfer to the stretching sheet.

Table 2 is plotted for the different values of Prandtl number Pr, Eckert number E and viscoelastic parameter k_1^* for wall temperature $g(0)$ in PHF case. Analysis of the tabular results shows that as the value of Prandtl number Pr increases for both $E = 0$ and $E \neq 0$ the wall temperature $g(0)$ decreases and increasing the values of viscoelastic parameter k_1^* leads to the increase of wall temperature $g(0)$.

6. Conclusions

A mathematical analysis has been carried out for momentum and heat transfer in a viscoelastic fluid flow over an exponentially stretching impermeable sheet. Highly non-linear differential equations are converted into a set of ordinary differential equations by applying similarity transformations and sequential solutions of the transformed momentum equation are obtained analytically by solving the non-linear Riccati type equation repeatedly. Zero-order approximate solution for stream function f is compared with the numerical solution obtained by employing Runge–Kutta fourth order method with shooting and desired accuracy has been achieved. Solutions for heat transfer equation are derived in the form of confluent hypergeometric function for both cases (i) prescribed surface temperature (PST) and (ii) prescribed boundary heat flux (PHF). Expressions are also obtained for dimensionless skin-friction coefficients C_f and Nusselt number Nu. The derived solutions involve an exponential dependence of stretching velocity, prescribed boundary temperature and prescribed boundary heat flux on the flow directional coordinate.

The important findings of the graphical analysis of the results of the present problem are as follows:

1. The effect of increasing the values of viscoelastic parameter k_1^* is to decrease the velocity throughout the boundary layer.
2. The effect of increasing the values of viscoelastic parameter k_1^* is to decrease the skin-friction parameter C_f and the effect of Reynolds number is also to decrease skin-friction coefficient C_f.
3. The effect of increasing the values of Prandtl number Pr is to decrease temperature distribution in the flow region.
4. The effect of increasing the values of viscoelastic parameter k_1^* is to increase the temperature distribution in the flow region.
5. The effect of increasing the values of Prandtl number Pr is to increase of wall temperature gradient $-\theta_\eta(0)$ whereas the effect of viscoelastic parameter k_1^* is to decrease wall temperature gradient $-\theta_\eta(0)$.
6. We can control heat transfer considerably by increasing the values of viscoelastic parameter k_1^* and Eckert number E. Significant increase of Eckert number might reverse the direction of heat transfer to the stretching sheet.

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